International Journal of Theoretical Physics, Vol. 35, No. 7, 1996

Wormhole Solution in Bergmann–Wagoner Scalar-Tensor Gravitational Theory

Xing-Guo Xiao[1] and Liao Liu[2]

Received May 8, 1995

We give a Euclidean wormhole solution in the vacuum Bergmann–Wagoner scalar-tensor gravitational theory. We show that this wormhole, unlike others, has complex charge and is a baby universe (half a wormhole).

1. INTRODUCTION

Bergmann–Wagoner gravitational theory (BWT) (Will, 1986) is a general scalar-tensor theory in which the Brans–Dicke parameter ω and cosmological function Λ depend upon the scalar gravitational field ϕ. The Brans–Dicke gravitational theory is a simple example of BWT corresponding to ω = const and $\Lambda = 0$.

Since Hawking (1988) found the first Euclidean wormhole solution in quantum cosmology, much work has been done in this field (Giddings and Strominger, 1988; Coleman, 1988; Ghoroka and Tamka, 1991). Because scalar-tensor theory is important for quantum cosmology, we seek the Euclidean wormhole solution of BWT. In this paper, on the basis of the wormhole wave function that satisfies Hawking's boundary condition in BWT quantum cosmology (Liu and Chen, 1992), the vacuum Euclidean wormhole of BWT is found. It is shown that its conserved charge is complex and it is a half wormhole (baby universe).

In Section 2 a Euclidean wormhole configuration is found by solving the Euclidean BWT equation; Section 3 contains a brief discussion.

[1] Department of Physics, Tianjin College of Education, Tianjin, 300020, China.
[2] Department of Physics, Beijing Normal University, Beijing, 100875, China.

0020-7748/96/0700-1503$09.50/0

2. WORMHOLE SOLUTION

The vacuum BWT action reads

$$S_{BW} = l_P^{-2} \int d^4x \sqrt{-g}[\phi R - \omega(\phi)\phi^{-1} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 2\phi\Lambda(\phi)] \tag{1}$$

for the $R^1 \times S^3$ topology with Roberts and Walker metric

$$ds^2 = \sigma^2(dt - a^2(t)\, d\Omega_3^2) \tag{2}$$

where $\phi(t)$ is the conventional real scalar gravitational field,

$$\sigma^2 = l_P^2/24\pi^2, \qquad l_P = \text{Planck length}$$

$$R = 6\sigma^{-2}a^{-2}(aa'' + a'^2 + 1) \tag{3}$$

and prime denotes the time derivative with respect to the cosmic time t.

After integration with respect to space coordinates and partial integration with respect to the term $a^2a''\phi$, we have

$$S_{BW} = \frac{1}{2}\int dt \left[-a'^2a\phi - a'\phi'a^2 + a\phi + \frac{\omega(\phi)}{6} a^3\phi^{-1}\phi'^2 - \frac{\sigma^2}{3} a^3\phi\Lambda(\phi) \right] \tag{4}$$

we introduce the transformations

$$\xi = \phi^{1/2}a, \qquad d\tau = \phi^{1/2}\, dt$$

$$\eta = \int \left(\frac{2\omega(\phi) - 3}{12}\right)^{1/2} \phi^{-1}\, d\phi, \qquad \lambda(\eta) = \frac{1}{3}\sigma^2\Lambda(\phi)\phi^{-1} \tag{5}$$

then

$$\frac{dt}{a} = \frac{d\tau}{\xi}, \qquad \frac{d\eta}{dt} = \left(\frac{2\omega(\phi) + 3}{12}\right)^{1/2} \phi^{-1}\frac{d\phi}{dt}$$

$$\frac{d\phi}{dt} = \phi^{3/2}\left(\frac{2(\omega(\phi) + 3)}{12}\right)^{-1/2} \frac{d\eta}{d\tau}$$

$$\frac{da}{dt} = \frac{d\xi}{d\tau} - \frac{1}{2}\xi\left(\frac{2\omega(\phi) + 3}{12}\right)^{-1/2} \frac{d\eta}{d\tau}$$

so the BW action becomes

$$S_{\mathrm{BW}} = \frac{1}{2}\int \frac{d\tau}{\xi}\left[\xi^2 - \xi^4\lambda(\eta) - \xi^2\left(\frac{d\xi}{d\tau}\right)^2 + \xi^4\left(\frac{d\eta}{d\tau}\right)^2\right] \tag{6}$$

let

$$d\tau = \xi(b)\, db, \qquad \frac{d\xi}{d\tau} = \xi^{-1}(b)\,\frac{d\xi}{db} \tag{7}$$

Introduce the conformal transformation

$$ds^2 = \xi^2(b)\sigma^2(-db^2 + d\Omega_3^2)$$

We have

$$S_{\mathrm{BW}} = \frac{1}{2}\int db\left[\xi^2 - \xi^4\lambda(\eta) - \left(\frac{d\xi}{db}\right)^2 + \xi^2\left(\frac{d\eta}{db}\right)^2\right] \tag{8}$$

Let

$$d\bar{b} = i\, db \tag{9}$$

We get the Euclidean action

$$S_{\mathrm{EBW}} = -iS_{\mathrm{BW}} = \frac{1}{2}\int d\bar{b}\left[\xi^2 - \xi^4\lambda(\eta) + \left(\frac{d\xi}{d\bar{b}}\right)^2 - \xi^2\left(\frac{d\eta}{d\bar{b}}\right)^2\right] \tag{10}$$

The corresponding Euclidean BW Lagrangian is (a dot denotes time derivative with respect to the Euclidean conformal time $\bar{b}$)

$$L_{\mathrm{EBW}} = \tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\xi^2 - \tfrac{1}{2}\xi^4\lambda(\eta) - \tfrac{1}{2}\xi^2\dot{\eta}^2 \tag{11}$$

The classical Euclidean BW gravitational equation can be derived from the least action principle.

By variation with respect to ξ and η, we get the ξ-equation

$$\frac{d}{d\bar{b}}\frac{\partial}{\partial\dot{\xi}}L - \frac{\partial}{\partial\xi}L = \ddot{\xi} - \xi + 2\xi^3\lambda(\eta) + \xi\dot{\eta}^2 = 0 \tag{12}$$

and the ξ-equation

$$\frac{d}{d\bar{b}}\frac{\partial}{\partial\dot{\eta}}L - \frac{\partial}{\partial\eta}L = \frac{d}{d\bar{b}}(-\xi^2\dot{\eta}) - \frac{1}{2}\xi^4\frac{\partial\lambda(\eta)}{\partial\eta} = 0 \tag{13}$$

Suppose that $\Lambda(\phi)$ is a slowly varying function of ϕ, which is to say, $\lambda(\eta)$ varies with η slowly. We can set

$$\frac{\partial\lambda(\eta)}{\partial\eta} \to 0$$

From (13) we obtain the conserved charge of the wormhole

$$q \equiv \xi^2\dot{\eta} \equiv a^3\left(\frac{2\omega(\phi) + 3}{12}\right)^{1/2} \frac{d\phi}{d\bar{t}} \tag{14}$$

where $\bar{t} = it$ is the Euclidean cosmic time.

After we substitute (14) in (12), we get

$$\ddot{\xi} - \xi + 2\xi^3\lambda(\eta) + \xi^{-3}q^2 = 0 \tag{15}$$

Now we define

$$f(\xi) = \dot{\xi}^2 - 1 \tag{16}$$

Then the second-order derivatives of $\xi(\bar{b})$ with respect to $\bar{b}$ are replaced by the first-order derivatives of $f(\xi)$ with respect to ξ:

$$\frac{df(\xi)}{d\xi} = 2\ddot{\xi} = 2\xi - 4\xi^3\lambda(\eta) - 2\xi^{-3}q^2 \tag{17}$$

After integration with respect to (17), we have

$$f(\xi) = \xi^2 - \xi^4\lambda(\eta) + \xi^{-2}q^2 + A \tag{18}$$

where A is an integration constant.

The criteria for $f(\xi)$ as a wormhole solution are for ξ to take its maximum value ξ_m and minimum value ξ_b equal to -1 and for $f(\xi)$ (≥ -1) to vary smoothly between ξ_m and ξ_b (Ghoroka and Tanuka, 1991).

When $f(\xi) = -1$, (18) becomes

$$\xi^6 - \lambda^{-1}\xi^4 - \lambda^{-1}(A + 1)\xi^2 - \lambda^{-1}q^2 = 0 \tag{19}$$

We let

$$\xi^2 = y + 1/(3\lambda)$$

Then (19) can be rewritten as

$$y^3 + Py + Q = 0 \tag{20}$$

where

$$P = -\frac{1 + 3\lambda(A + 1)}{3\lambda^2}, \qquad Q = -\frac{2}{27\lambda^3}\left[1 + \frac{9}{2}\lambda(A + 1)\frac{27}{2}\lambda^2q^2\right]$$

From the criteria demands for a wormhole solution we have $\xi_m >> \xi_b$ and $\xi^2 > 0$, so the roots y_1 and y_2 (corresponding to ξ_m and ξ_b, respectively) of equation (20) should satisfy

$$y_1 >> y_2, \qquad y_{1,2} > -1/(3\lambda) \tag{21}$$

According to the criterion for a cubic equation, when

$$\Delta = \left(\frac{Q}{2}\right)^2 + \left(\frac{P}{3}\right)^3 \geq 0$$

equation (19) does not have a wormhole solution, owing to the fact that the roots of equation (20) do not satisfy condition (21). When

$$\Delta < 0$$

equation (20) has three nonequivalent real roots, so equation (19) may have a wormhole solution. We rewrite (20) as

$$y - my^3 = l \tag{22}$$

where

$$l = -\frac{2}{9\lambda}[1 + 3\lambda(A + 1)]^{-1}\left[1 + \frac{9}{2}\lambda(A + 1) + \frac{27}{2}\lambda^2 q^2\right], \qquad m = \frac{3\lambda^2}{1 + 3\lambda(A + 1)} \tag{23}$$

By using a graphical method we can evaluate the approximate solution of equation (22). In Fig. 1 we give two curves to indicate

$$\eta = y - my^3 \tag{24}$$

$$\eta = l < 0 \tag{25}$$

respectively. The intersection points y_1, y_2, y_3 in Fig. 1 give the solution of equation (22).

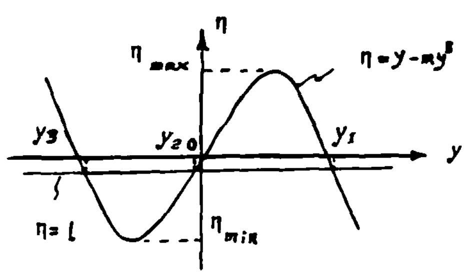

Fig. 1.

When

$$y = \pm\frac{\sqrt{1 + 3\lambda(A + 1)}}{3\lambda} \tag{26}$$

equation (24) has extreme values

$$\eta_{\max} = \frac{2}{9\lambda}\sqrt{1 + 3(A + 1)\lambda}, \qquad \eta_{\min} = -\frac{2}{9\lambda}\sqrt{1 + 3(A + 1)\lambda} \tag{27}$$

Supposing $|l| << \eta_{\max}$, then only if

$$A + 1 >> 1/(3\lambda) \tag{28}$$

can this condition be satisfied, and solving equation (22) becomes easier. It is observed from Fig. 1 that the root y_2 of (22) is very small; the y^3 term can be neglected from equation (22), and we have

$$\begin{aligned} y_2 &\approx l \\ &= -\frac{2}{9\lambda}[1 + 3\lambda(A + 1)]^{-1}\left[1 + \frac{9}{2}\lambda(A + 1) + \frac{27}{2}\lambda^2 q^2\right] \\ &\approx -\frac{q^2}{A + 1} - \frac{1}{3\lambda} \end{aligned} \tag{29}$$

In order to meet condition (21), we must have $-q^2/(A + 1) > 0$, so q should be imaginary, i.e.,

$$q = i|q| \tag{30}$$

where $|q|$ is real. Then (29) becomes

$$y_2 = \frac{|q|^2}{A + 1} - \frac{1}{3\lambda} \tag{31}$$

Obviously, the root y_1 of equation (22) is large, so the l term can be neglected. We have

$$y_1 = \frac{\sqrt{1 + 3\lambda(A + 1)}}{\sqrt{3}\lambda} \approx \frac{\lambda(A + 1)}{\lambda} \tag{32}$$

However, because

$$y_3 < -\frac{\sqrt{1 + 3\lambda(A + 1)}}{3\lambda}$$

does not satisfy condition (21), the root y_3 can be ignored. So from

$$\xi_b^2 = y_2 + \frac{1}{3\lambda} = \frac{|q|^2}{A+1}, \qquad \xi_m^2 = y_1 + \frac{1}{3\lambda} \approx \frac{1 + 3\sqrt{\lambda(A+1)}}{3\lambda}$$

we obtain

$$a_b = \frac{|q|}{\sqrt{(A+1)\phi}}, \qquad a_m = \left(\frac{1 + 3\sqrt{\lambda(A+1)}}{3\lambda\phi}\right)^{1/2} \tag{33}$$

Since $A + 1$ ($>> 1/3\lambda$) is very large and λ in general is very small, we have

$$a_b << a_m$$

Obviously, $a_b = l$ is the minimum dimension of the configuration and hence it is called the throat of the wormhole.

If we demand that a_b is not smaller than the Planck length l_P, then (33) give

$$\frac{1}{3\lambda} << A + 1 \leq \frac{|q|^2}{l_P^2\phi}$$

This condition can certainly be fulfilled.

Therefore, the above solution (33) is just the Euclidean wormhole configuration connecting a baby universe (with the size of the Planck length $\sim a_b$) and a large mother universe (with a size of a_m).

3. DISCUSSION

1. No Euclidean wormhole solution has been found for the vacuum Einstein field equation, but we have found a wormhole solution for the BWT without other matter fields. Moreover, the Euclidean wormhole configuration is also different from that of Hawking (1988), Giddings and Strominger (1988), and Coleman (1988). It is a half wormhole, also called a baby universe.

2. From the definition of conserved charge (14), the complex conserved charge (imaginary number charge) demands $\omega(\phi) < -3/2$, i.e., $\omega(\phi)$ should be a negative number less than $-3/2$. We note that solar system experiments now only constrain the coupling constant ω to $|\omega| > 500$ (Will, 1981), namely $\omega > 500$ or $\omega < 500$, so our result is permitted by the experiments.

3. Our result rests on the supposition of $\lambda(\phi)$ varying slowly with ϕ. If this is not the case, we cannot find a wormhole solution for BWT.

REFERENCES

Coleman, S. (1988). *Nuclear Physics B*, **310**, 643.
Ghoroka, K., and Tanuka, M. (1991). *Physical Review D*, **43**, 410.
Giddings, S. B., and Strominger, A. (1988). *Nuclear Physics B*, **306**, 890.
Hawking, S. W. (1988). *Physical Review D*, **37**, 904.
Liu, Z. X., and Chen, X. J. (1992). *Chinese Physics Letters*, **9**, 673.
Will, C. M. (1981). *Theory and Experiment in Gravitational Physics*, Cambridge University Press, Cambridge, p. 206.